

Designing a flippe top

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September 2020

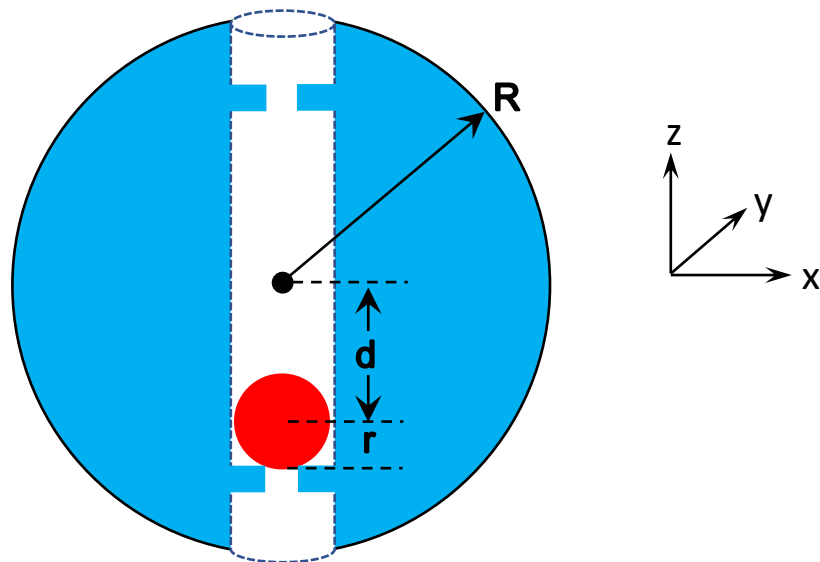


Figure 1: Diagram of a prototype flippe top.

Figure 1 shows a design for a new kind of spinning top. We start with a solid sphere of radius R . A hole of radius r is bored through the center and a steel ball (of radius r) is inserted. A stop at each end is necessary to keep the steel ball inside the sphere. The maximum distance the steel ball (center) can move away from the sphere center is denoted by d . At this stage there are only three design parameters: R , r and d , although the density of the materials used is also important.

Our goal is to design this top so that it behaves like a classical tippe top. When spinning at a sufficiently high speed, the classical tippe top is defined by two properties:

- 1) The Figure 1 position when the center of mass is directly below the center of rotation is **unstable**.
- 2) The “inverted state” where the center of mass is directly above the center of rotation is **stable**.

Suppose we can design the Figure 1 top with these two properties. When this top is spun about the z -axis, it is an unstable position and will invert. After inversion the steel ball will (presumably) drop down; the top is back at the starting position, but flipped 180° . This process will repeat, flipping over and over. Because this is a tippe top which flips repeatedly, I call it a “flippe top”.

How can we determine if the top in Figure 1 will behave like a classical tippe top? Fortunately, several recent papers [1, 2] have considered the behavior of a sphere with an offset center of gravity. These papers give specific criterion for when the Figure 1 state is unstable and the inverted state is stable.

Material	Density (g/cm^3)
Hardwood	0.75
Stainless steel	7.8
PLA	1.25
PETG	1.27
ABS	1.07

Table 1: Top material densities.

We denote by I_x the moment of inertia of the top about the x-axis and by I_z the moment of inertia about the z-axis (we assume the top is symmetric in the x-y plane, so that $I_x = I_y$). The criterion for “tippe top behavior” is complex, but as an initial design target we want

$$I_z > I_x$$

All changes we make to the sphere can be thought of either as helping this inequality or going against it. For example, the steel ball increases I_x and only slightly increases I_z , so acts against this inequality. The hole through the center helps this inequality. This is one reason that we choose to bore the hole through the entire sphere. Such a large hole may affect the spinning dynamics when $r/R > 1/4$ —in that case end caps may be necessary.

A wood flippe top

The easiest way to create a wood flippe top is to start with a solid wood sphere and cut a hole through the center. We then add a steel ball and a pair of stops to trap it. For stops one could use a pair of wooden washers. To determine good values for R, r and d we need to calculate the moment of inertia of this object.

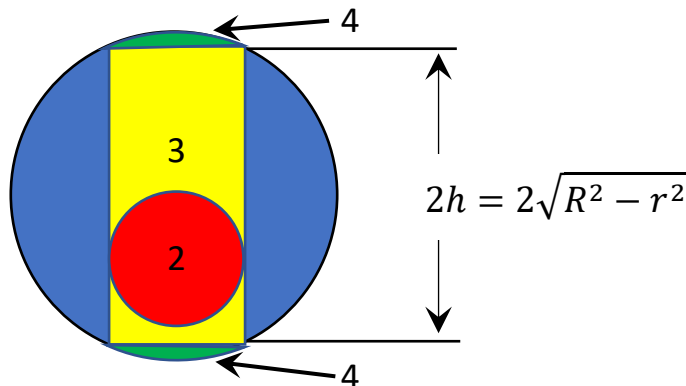


Figure 2: The four components of a wood flippe top. Component 1 is the full sphere.

Let us now calculate $I_x = I_y$ and I_z . Basically, we split the top into four components, shown in Figure 2. The mass and moments of inertia of each component are calculated, then added up to give the total mass and moments of inertia. If the mass of a component is negative it means that this component is removed.

Component 1 is a solid wood sphere with density $\rho_{wood} = 0.75 \text{ g/cm}^3$. The density of wood varies, but 0.75 g/cm^3 is a reasonable estimate for hardwood. I have many 1" diameter Maple balls which I weighed at 6.3 g each, this gives a density of 0.73 g/cm^3 .

For component 1, we have

$$m_1 = \frac{4}{3}\pi R^3 \rho_{wood}$$

$$I_{1x} = I_{1z} = \frac{2}{5}m_1 R^2$$

Here m_i , I_{ix} and I_{iz} represent the mass and moments of inertia of component i . We will also use c_{iz} for the z-coordinate of the center of mass of component i , in this case c_{1z} is zero. In what follows, if c_{iz} is not defined it is zero.

Component 2 is the steel ball,

$$m_2 = \frac{4}{3}\pi r^3 \rho_{Fe}$$

$$I_{2x} = I_{2z} = \frac{2}{5}m_2 r^2$$

$$c_{2z} = -d$$

Component 3 (to be removed) is a cylinder of height $2h$, radius r , and density $-\rho_{wood}$, where $h = \sqrt{R^2 - r^2}$

$$m_3 = -2\pi r^2 h \rho_{wood}$$

$$I_{3x} = \frac{m_3}{12}(3r^2 + 4h^2)$$

$$I_{3z} = \frac{m_3}{2}r^2$$

Component 4 (to be removed) are two cylindrical caps between $z=-R$ and $-h$, and $z=h$ and R where $h = \sqrt{R^2 - r^2}$. This component is very small when $r \ll R$ and can be ignored. This component is by far the most complex to calculate, but has almost no effect on the results. These formulas were verified using Mathematica. They also give the same values as the solid sphere when $r=R$ and $h=0$.

$$m_4 = -\frac{2\pi\rho_{wood}}{3}(R-h)^2(2R+h) = -\left(m_1 + m_3 - \frac{4\pi}{3}h^3\rho_{wood}\right)$$

$$I_{4x} = \frac{m_4}{20} \frac{16R^3 + 17R^2h + 18Rh^2 + 9h^3}{2R+h}$$

$$I_{4z} = \frac{m_4}{10} \frac{(R-h)(8R^2 + 9Rh + 3h^2)}{2R+h}$$

“napkin ring” term

The total mass of the top is $m_t = \sum_1^4 m_i$. The center of mass of the top has z-coordinate ϵ , where

$$\epsilon = \sum_{i=1}^n c_{iz} \frac{m_i}{m_t} = -d \frac{m_2}{m_t}$$

By the parallel axis theorem, the total moments of inertia about the center of mass are given by a sum over the $n = 4$ components:

$$I_x = \sum_{i=1}^n I_{ix} + m_i (c_{iz} - \epsilon)^2$$

$$I_z = \sum_{i=1}^n I_{iz}$$

These values can then be used to create a phase diagram for the wood flippe top.

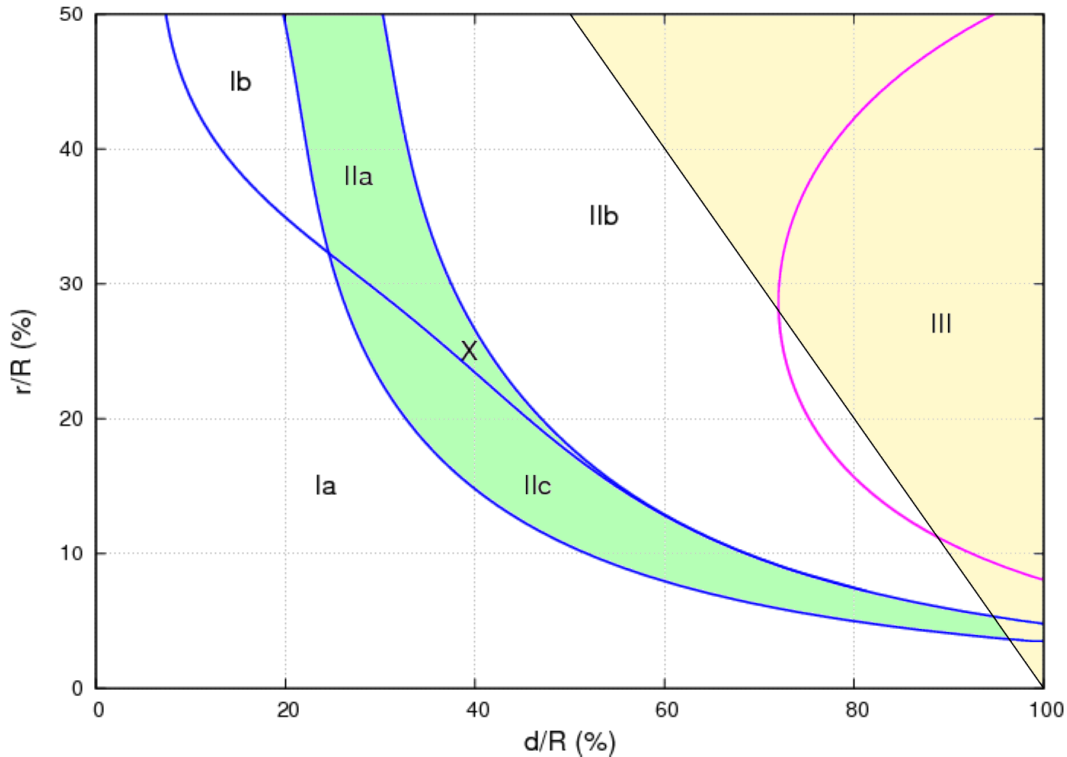


Figure 3: Phase diagram of the wood flippe top (r/R vs. d/R), see [1] for group definitions. A well-behaved flippe top must lie in the middle green strip.

Figure 3 shows the phase diagram for the wooden flippe top. The Roman numerals identify Groups as defined in [1], “classical tippe tops” lie in Group II. Tops in group IIb have stable “intermediate states”, as defined in [1]. While these may not be a problem for a tippe top that flips only once, the flippe top must invert many times over a wide range of rotational velocities. We don’t want the top to stop at an intermediate state, so a flippe top design should aim for Group IIa or Group IIc, the green region in Figure 3. For any choice of r/R this gives a narrow range of d values for the flippe top.

The yellow region in Figure 3 indicates $d + r > R$. These tops are problematic because the steel ball extends beyond the large ball and may interfere with spinning.

For example, suppose we start with a 2" diameter hardwood ball, $R=25.4$ mm and weight 51.5 g. A reasonable value for r/R is $1/4$ so we select a $1/2$ inch diameter steel ball. From Figure 3 we find that a flippe top with $r/R = 25\%$ lies in Group IIa or IIc when $28.4\% < d/R < 41.4\%$. Thus, we should use a value of d between 7.2 mm and 10.5 mm, with $d=10$ mm marked by the 'X' in Figure 3.

One important parameter not shown in Figure 3 is the time it takes for the top to invert. We can infer, however, that the inversion time goes to infinity at the border between groups II and I, so we want to be at the high end of the range $28.4\% < d/R < 41.4\%$.

Note that Figure 3 depends on the hardwood density used, as well as the density of the steel ball. If wood of a different density is used, or the ball is 3D printed, the curves shift to the right or left. The wood flippe top is scale invariant. If a wood ball of twice the size is used, it should behave the same provided all other lengths are also doubled (the steel ball must also be twice as large and eight times heavier).

A 3D printed flippe top

Objects 3D printed using an FDM (Fused Deposition Modeling) printer are generally not solid. It is faster to print the interior at a much lower density, the percent of interior which is filled is called the **infill rate** (f). In order to print faster this fraction is usually as small as possible. A typical value is $f=15\%$, but in designing a 3D printed flippe top we can use any value between 10% and 100%.

Each layer needs to be supported by the layers below, making a sphere difficult to print. One can add "support material" which is eventually discarded, but a better option is to cut the sphere in half and print it in two parts. Figure 4 shows a half-flippe top ready for 3D printing. After two copies are printed, they are simply glued together and/or held together with metal pins for alignment. The pins help hold the halves together, and also keep the central channel aligned. If you would like to print a copy, design files can be found at [3].

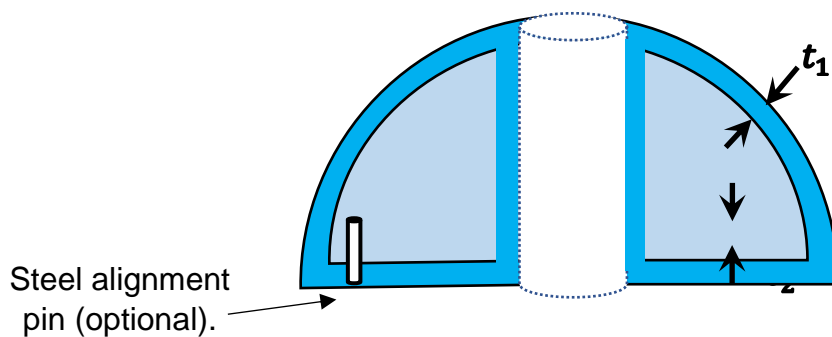


Figure 4: A design for half a flippe top for 3D printing.

In addition to R , r and d we now have two thickness parameters t_1 and t_2 , as well as the fill ratio f . In general, if the ball is scaled up, the thickness parameters stay the same. For this reason, the 3D printed flippe top is not scale invariant, although it is approximately so.

We can calculate the moments of inertia of the top in Figure 4, but we now have nine components:

- 1) A solid sphere of spherical shell of density ρ . We will use $\rho = \rho_{PLA}$.
- 2) The steel ball of radius r and z -coordinate $-d$.
- 3) Remove a solid sphere of radius $R - t_1$ and density $-\rho(1 - f)$.
- 4) Remove a cylinder of radius r , height $2(R - t_1)$ and density $-f\rho$.
- 5) Remove the “top lid”, a cylinder of radius r , height t_1 and density $-\rho$.
- 6) Remove the “bottom lid”, a cylinder of radius r , height t_1 and density $-\rho$.
- 7) Add back the “inner wall”, a hollow cylinder with outer radius $r + t_1$, inner radius r , height $2(R - t_1)$ and density $\rho(1 - f)$.
- 8) Add the “base plate” (doubled), a hollow cylinder with outer radius $R - t_1$, inner radius $r + t_1$, height $2t_2$ and density $\rho(1 - f)$.
- 9) (optional) Add several steel alignment pins (see Figure 4). These hold the top together but also increase I_z .

We note that components 4-8 are not exact, but assume $t_i \ll R$. The phase diagram of the PLA tippe top with variable infill is shown in Figure 5. The $f=100\%$ top line in Figure 5 should correspond to the $r/R=0.254$ line in Figure 3. It doesn't match because PLA has a higher density than wood.

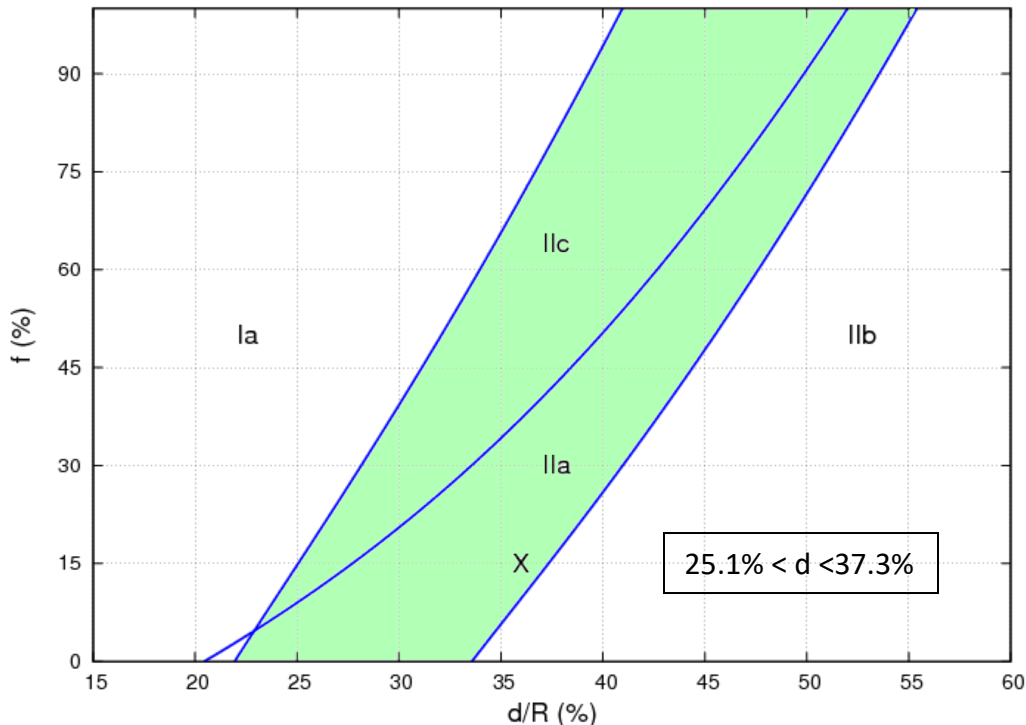


Figure 5: Phase diagram of the 3D printed flippe top (f vs. d/R), $r/R=0.254$.

I made many copies of the flippe top with the parameters in Table 2 (design files are available at [3]). In addition, I also scaled the entire top up by a factor of 5/4 and down by a factor of 3/4. All of them are able to execute multiple flips, as many as seven flips in the case of the smallest version. Figure 6 shows all three sizes.

Parameter	Value
R	25.0 mm
r	6.35 mm = 1/4 inch
t ₁	0.9 mm (2 perimeters)
t ₂	1.1 mm (7 layers)
f	15%
d	9 mm (d/R = 36%)
ρ	1.25 g/cm ³ (PLA)

Table 2: Flippe top parameters.

Table 3 shows specifications for all three sizes of flippe top in Figure 6. The measured weight of each top is within 1 g of the Table 3 values, giving us confidence in our model.



Figure 6: Three sizes of 3D printed flippe tops. The largest one has an equatorial groove for a string.

Each time the top flips, some of its rotational kinetic energy is used to raise the steel ball by an amount $2d$. Starting the top spinning by hand there is a limit to the initial velocity of the top, in the neighborhood of $v_{max} \approx 125 \text{ cm/s}$ speed of the top circumference. If all the top's rotational energy is used to raise the ball, the maximum number of flips which can be executed is

$$n_{max} = \frac{\frac{1}{2} I_z \omega^2}{m_2 g (2d)} = \frac{I_z v_{max}^2}{4 m_2 g R^2 d}$$

Inserting values for each size top gives the values for n_{max} given in Table 3. We find that smaller top and the wood top are capable of the most flips. I have uploaded two movies of tops in action on youtube [4, 5].

Parameter	Base model	5/4 scale	3/4 scale	Wood version
2R (ball diameter)	5 cm	6.25 cm	3.75 cm	2 inch
2r (steel ball diam)	1/2 inch	5/8 inch	3/8 inch	1/2 inch
d	9 mm	11.25 mm	6.75 mm	10 mm
m_2 (steel ball weight)	8.4 g	16.3 g	3.5 g	8.4 g
m_f (total weight)	32.6 g	58.5 g	15.7 g	55.1 g
ϵ/R (eccentricity)	9.2%	10.1%	8.1%	2.0%
$I_x = I_y$	73.0 $g\ cm^2$	199.6 $g\ cm^2$	20.3 $g\ cm^2$	124.6 $g\ cm^2$
I_z	75.5 $g\ cm^2$	204.4 $g\ cm^2$	21.4 $g\ cm^2$	133.3 $g\ cm^2$
printing time	2.8 hours	4.3 hours	1.5 hours	years to grow!
n_{max}	6.4	4.5	10.1	9.8

Table 3: Flippe top specifications for different sizes.

In order to get a faster initial rotation, a string can be wrapped around the top.

Observations of high-speed tops show a new phenomenon: the top performs one flip, but then the steel ball does not drop down (see the video [5]). The inverted top then spins until the rotational velocity drops below a critical value when the steel ball drops and flipping continues. It would be interesting to calculate this critical rotational velocity.

Summary

We have calculated design parameters for a (solid) wood flippe top and a 3D printed flippe top. I have printed many sizes of the 3D printed version, using many different values for d as well as the infill percentage f. The best versions use parameters given in Tables 2 and 3.

References:

- 1) M.C. Ciocci, B. Malengier, B. Langerock and B. Grimonprez, "Towards a Prototype of a Spherical Tippe Top", *Journal of Applied Math* doi:10.1155/2012/268537
- 2) M. C. Ciocci and B. Langerock, "Dynamics of the tippe top via Routhian reduction", *International Journal of Bifurcation and Chaos*, V 12, no. 6, pp.602-14, 2007.
- 3) <https://www.thingiverse.com/thing:3990145>
- 4) <https://www.youtube.com/watch?v=aO6ZofZ5dos> (movie of a small flippe top)
- 5) https://www.youtube.com/watch?v=3E_Ffhsj5ml (movie of a regular flippe top)